HEAT FLUX MEASUREMENT BY THE TEMPERATURE FIELD OF A HEATED SURFACE. 1. UNIFORM FLUX

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Calculation formulas are suggested for solving the three-dimensional inverse problem of reconstruction of the heat flux by the temperature of a heated surface. For the special case where the heat flux is uniformly distributed over the heating surface, a numerical experiment on reconstruction of the heat flux is performed for various boundary conditions of heating.

In solving heat transfer problems in hydrodynamics, aerodynamics, a number of technological processes, and laser treatment of materials, one is faced with the problem of determination of the heat flux by the temperature of heated surfaces. The corresponding inverse problem pertains to the class of conventionally correct problems and is called the problem of recalculation of boundary conditions [1].

In [2], analytical solutions of the nonstationary problem of recalculation of boundary conditions are obtained for the situation where heat transfer proceeds on a face of a uniform plate infinitely extended in the transverse direction. The back surface of the plate may be heat-insulated or maintained at a constant temperature. In the present work the corresponding analytical representations [2] are realized in the form of algorithms for the case where a heat flux incident onto a plate is uniform on its surface. In a numerical experiment the authors investigate the quality of the reconstruction.

A solution of the problem of recalculation of boundary conditions is described in detail in [2] for a semi-infinite solid and for a plate of finite thickness with a cooled or heat-insulated back surface. We present the corresponding relations, assuming that the initial temperature is equal to zero on the entire plate.

a) The back surface of the plate is maintained at a constant temperature ($T(0, \boldsymbol{\rho}, t) = 0$). In accordance with [2] we write

$$q(\boldsymbol{\rho}, t) = -\frac{\pi k}{a^2 L}\int_0^t d\tau \frac{d}{d\tau}\vartheta_1\left(1/2, \frac{t-\tau}{L^2}a^2\right) \times \iint_{-\infty}^{\infty} d\boldsymbol{\rho}' \frac{T(t, \boldsymbol{\rho}')}{(t-\tau)} \exp\left\{-\frac{(\boldsymbol{\rho}'-\boldsymbol{\rho})^2}{4a^2(t-\tau)}\right\}. \quad (1)$$

Relation (1) is valid for arbitrary values of a^2, L, and the variables $\boldsymbol{\rho}$ and t. However, it may be simplified provided certain relationships exist between these quantities.

For Fo >> 1, from Eq. (1) we derive the asymptotic formula

$$q(t, \boldsymbol{\rho}) = \frac{k}{L} T(t, \boldsymbol{\rho}) + \frac{L^2 k}{3a^2}\left\{\frac{\partial T(t, \boldsymbol{\rho})}{\partial t} - a^2 \Delta_\perp T(t, \boldsymbol{\rho})\right\}. \quad (2)$$

b) In an approximation of a semi-infinite solid (Fo << 1) Eq. (1) is transformed into an equation with an Abel kernel:

$$q(t, \boldsymbol{\rho}) = \frac{1}{2a^2\sqrt{\pi^3}}\int_0^t d\tau \iint_{-\infty}^{\infty} d\boldsymbol{\rho} \frac{T(\tau, \boldsymbol{\rho}) \exp\left(-\frac{(\boldsymbol{\rho}'-\boldsymbol{\rho})^2}{4a^2(t-\tau)}\right)}{\sqrt{(t-\tau)^5}}. \quad (3)$$

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c) If the back surface of the plate is heat-insulated ($(\partial T/\partial z)(0, \boldsymbol{\rho}, t) = 0$), the flux is related to the temperature on the target surface as

$$q(\boldsymbol{\rho}, t) = -\frac{\pi k}{a^2 L}\int_0^t \tau \frac{d}{d\tau}\vartheta_3\left(1, \frac{t-\tau}{L^2}a^2\right) \times \iint_{-\infty}^{\infty} d\boldsymbol{\rho}' \frac{T(t, \boldsymbol{\rho}')}{(t-\tau)} \exp\left(-\frac{(\boldsymbol{\rho}'-\boldsymbol{\rho})^2}{4a^2(t-\tau)}\right). \tag{4}$$

In the case Fo >> 1, from (4) it follows that

$$q(t, \boldsymbol{\rho}) = \frac{Lk}{a^2}\left\{\frac{\partial T(t, \boldsymbol{\rho})}{\partial t} - a^2 \Delta_\perp T(t, \boldsymbol{\rho})\right\}, \tag{5}$$

and at Fo >> 1 representation (4) turns into (3) with an Abel kernel in the time coordinate.

We shall investigate solutions for the case where the flux is distributed uniformly over the plate face. It is obvious that $T(z, \boldsymbol{\rho}, t) = T(z, t)$. Instead of (1), (2), (4) and (5) we arrive at

$$q(t) = -\frac{2k}{L}\int_0^t \frac{dT(\tau)}{d\tau}\vartheta_1\left(1/2, \frac{t-\tau}{L^2}a^2\right) d\tau. \tag{6}$$

$$q(t) = -\frac{kL}{a^2}\frac{dT(t)}{dt}, \tag{7}$$

$$q(t) = -\frac{k}{a\sqrt{\pi}}\int_0^t \frac{dT(\tau)\, d\tau}{d\tau\sqrt{t-\tau}}, \tag{8}$$

$$q(t) = -\frac{k}{L}\int_0^t \frac{dT(\tau)}{d\tau}\vartheta_3\left(1, \frac{t-\tau}{2}a^2\right) d\tau, \tag{9}$$

$$q(t) = -\frac{k}{L}\left[T(t) - T_H + \frac{1}{3}\frac{L^2}{a^2}\frac{dT(t)}{dt}\right]. \tag{10}$$

We now give solutions of the direct problem for cases a), b), c), respectively:

$$T(t) = -\frac{2a^2}{kL}\int_0^t q(\tau)\vartheta_3\left(1, \frac{t-\tau}{2}a^2\right) d\tau. \tag{11}$$

$$T(t) = -\frac{a}{k\sqrt{\pi}}\int_0^t q(\tau)\frac{d\tau}{\sqrt{t-\tau}}. \tag{12}$$

$$T(t) = -\frac{a^2}{kL}\int_0^t q(\tau)\vartheta_1\left(1/2, \frac{t-\tau}{L^2}a^2\right) d\tau. \tag{13}$$

The solution for a heat-insulated plate coincides with that obtained earlier in [3]. Formulas (6), (8), (9) may be rewritten in a form free of temperature differentiation:

$$q(t) = -\frac{k}{L}T(t)\vartheta_3\left(1, \frac{t}{2}a^2\right) +$$

$$+ \int_0^t d\tau \, [T(t) - T(\tau)] \frac{d}{d\tau} \vartheta_3 \left(1, \frac{t-\tau}{L^2} a^2 \right), \quad (T(0, t) = 0), \tag{14}$$

$$q(t) = -\frac{k}{a\sqrt{\pi}} \left\{ \frac{T(t)}{\sqrt{t}} + \frac{1}{2} \int_0^t \frac{T(t) - T(\tau)}{(t-\tau)^{3/2}} d\tau \right\}, \quad \mathrm{Fo} \ll 1, \tag{15}$$

$$q(t) = -\frac{2k}{L} T(t) \vartheta_1 \left(1/2, \frac{t}{L^2} a^2 \right) +$$

$$+ \int_0^t d\tau \, [T(t) - T(\tau)] \frac{d}{d\tau} \vartheta_1 \left(1/2, \frac{t-\tau}{L^2} a^2 \right) d\tau, \quad \left(\frac{\partial T}{\partial z}(0, t) = 0 \right). \tag{16}$$

The structure of the integrands in relations (6), (8), (9), (14)-(16) is such that integration fails to smooth a noise component entering the measured temperatures. Therefore direct calculation of $q(t)$ by these formulas leads to instability of conversion [4]. Therefore in developing the corresponding algorithms for reconstruction the function $T(t)$ has been approximated by smoothing cubic splines [4] that take into account the level of the measurement error.

Reconstruction of the intensity for thermophysical situations a) and b) was modeled in a numerical experiment. Aluminum was chosen as the material of the plate. It was assumed that temperature values are recorded at the points τ_i: $0 = \tau_1 < \tau_2 < \tau_3 < \ldots < \tau_n = 1$ sec with the time resolution $\Delta t = 1/24$ sec. The initial data had a random measurement error ζ_i so that

$$T_i = T(\tau_i) + \zeta_i, \quad i = \overline{1, n}.$$

It was assumed that ζ_i follows the normal distribution law with the zero mean and dispersion σ^2. The function $T(t)$ was approximated by a smoothing cubic spline $S_{n,\alpha}(t)$ that takes the boundary conditions $S''_{n,\alpha}(0) = 0$, $S''_{n,\alpha}$ ($t =$ 1 sec) $= 0$ [4] taking into consideration the specific features of the solution of incorrect problems. With this substitution the integrals in the right-hand sides of (6), (8), (9) and (14)-(16) are calculated analytically. Infinite summation in formulas (6), (9), (14), (16) was restricted by the specified error. In all cases it did not exceed 1%.

For the numerical experiment we chose two functions as the models of the initial heat flux: the "pulse"

$$q(t) = I_0 \Theta(\tau) \left\{ 17\tau^2 - 32\tau^3 + 14\tau^4 + 1 \right\}, \quad \tau \le 1, \quad \Theta(\tau) = \begin{cases} 1, & \tau \ge 0, \\ 0, & \tau < 0, \end{cases} \tag{17}$$

where $I_0 = 10$ W/cm^2, $\tau = t/t_0$, $t_0 = 1$ sec, and the "cap"

$$q(t) = \begin{cases} I_0 \exp\left\{ \dfrac{(\tau - 0.5)^2}{(0.5)^2 - (\tau - 0.5)^2} \right\}, & |\tau - 0.5| \le 0.5, \\ 0, & |\tau - 0.5| > 0.5. \end{cases} \tag{18}$$

Dependences (17), (18) are shown in Figs. 1 and 2 (curves 1). In the absence of errors in the measured temperatures the error in calculating $q(t)$ is attributable to substitution of the cubic spline $S_n(t)$ for $T(t)$ and of finite summation for infinite summation. As the numerical experiment revealed, such an approximation provides sufficiently high accuracy of calculation of the corresponding integrals. Even in the case of "pulse" dependence (17) for $\Delta t = 1/24$ the error of reconstruction of $q(t)$ does not exceed 2.5%. In the solution reconstructed with the use of interpolation splines in the presence of measurement noise (curves 2 in Figs. 1, 2) random oscillations emerge that increase with the integration interval, and therefore we have used the smoothing splines $S_{n,\alpha}(t)$ in the subsequent analysis. The smoothing parameter α was chosen by the discrepancy method [4].

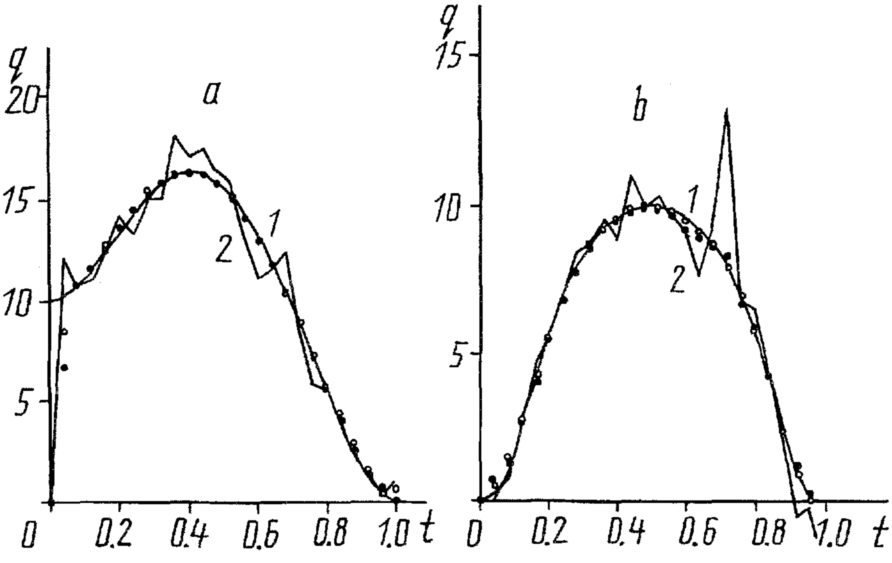

Fig. 1. Restoration of the model dependences $q(t)$ by the temperature of the cooled target surface: a) by formula (17); b) by formula (18); 1) exact solution; 2) restored solution with use of splines; dark points) smoothed solution based on expression (6); light points) smoothed solution based on (14). q, W/cm^2; t, sec.

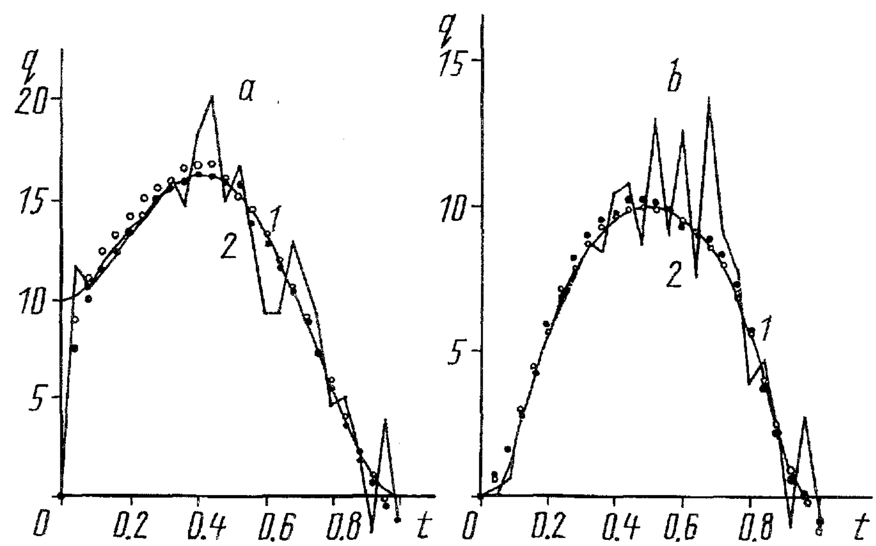

Fig. 2. Restoration of the model dependences $q(t)$ by the temperature of the heat-insulated target surface: a) by formula (17); by formula (18); 1) exact solution; 2) reconstructed solution with the use of splines; dark points) smoothed solution based on expression (9); light points) smoothed solution based on (16).

Figures 1 and 2 show results of reconstruction of $q(t)$ by model dependences (17) and (18) with an error of 3% in the input data. For model dependence (18) $q(t)$ is reconstructed more exactly. Figure 1 corresponds to a cooled target, and Fig. 2 to a heat-insulated one. Dark points show results of reconstruction using dependences (6), (9), and light points pertain to results calculated by (14), (16). As is seen in Fig. 1, the quality of reconstruction of $q(t)$ depends weakly on the type of boundary-value problem. The dependences reconstructed by formulas (14), (16), which contain no differentiation under the integral sign, are more exact. For comparison, we calculated the "dispersion" of the measurement error for a given t on the basis of the volume sampling N_{samp} (N_{samp} = 10):

TABLE 1

Time	The solution at $\sigma = 0.03$ for the boundary conditions			
	$dT(0, t) dz = -(1/k) q(t)$; $T(L, t) = 0$		$dT(0, t)/dz = -(1/k) q(t)$; $dt(0, t)/dz = 0$	
	Error "dispersion" of the solution based on the algorithms			
	(17)	(23)	(19)	(25)
Δt	0.141	0.021	0.1657	0.031
$2\Delta t$	0.038	0.003	0.050	0.004
$3\Delta t$	0.011	0.003	0.017	0.002
$4\Delta t$	0.004	0.003	0.007	0.002
$5\Delta t$	0.002	0.002	0.004	0.002
$D^2(q_{n,\alpha})$	0.0006	0.0002	0.0008	0.0002

$$\Delta^2(t) = \frac{1}{N_{\text{samp}}} \sum_{l=1}^{N_{\text{samp}}} (q_{n,\alpha}^{(l)}(t) - q(t))^2 . \tag{19}$$

Here $q(t)$ is the exact solution; $q_{n,\alpha}^{(l)}(t)$ is the solution constructed by the l-th realization of the noise values of temperature on the basis of smoothing splines with the smoothing parameter α.

Results of this comparison for $\sigma = 0.03$ in the case of "pulse" dependence (17) are presented in Table 1, where

$$D(q_{n,\alpha}) = \left[\frac{1}{n_t} \sum_{j=1}^{n_t} \Delta^2(t_j) \right]^{1/2} \tag{20}$$

is the root mean square error of the solution ($n_t = 25$).

It may be inferred from Table 1 that for algorithms without differentiation of the measured temperature values, $\Delta^2(t)$ may be substantially smaller than the corresponding values for algorithms with differentiation. A comparison of calculations performed for the "pulse" and the "cap" shows that such a difference is typical for the initial sections of "pulse" dependence (17). For the smoother initial function (18) the difference is less pronounced. Numerical experiments were conducted for targets with $L = 1$ cm. For the observation time $t = 1$ sec the Fourier parameter Fo was 0.86. Provided the condition Fo ≤ 0.2 is fulfilled, reconstruction may be accomplished using algorithms based on the inverse Abel transformation (8), (15).

To sum up, we have investigated reconstruction of the heat flux by the temperature of the heated surface of a plate infinitely extended in the transverse direction. Analytical solutions obtained ealier [2] were investigated numerically for a situation where the heat flux is uniformly distributed over the heated surface. The accuracy of algorithms for typical models of a time-dependent intensity was evaluated and it was shown that use of smoothing splines and algorithms without differentiation of temperatures improves the accuracy of the reconstruction.

NOTATION

$T(t, \boldsymbol{\rho})$, temperature on the plate surface; $q(t, \boldsymbol{\rho})$, heat flux on the plate surface; $\boldsymbol{\rho} = \{x, y\}$, transverse coordinate; t, time; K, thermal conductivity; a^2, thermal diffusivity; L, plate thickness; $\vartheta_1(\eta, \xi) = 2\sum_{k \in z} (-1)$ $\exp(-\pi^2(k - 1/2)^2)$ $\sin[\pi\eta(2k + 1)]$, Jacobi theta function [5]; Fo $= a^2 t^2/L^2$, Fourier parameter; $\vartheta_3(\eta, \xi) =$

$= \sum_{k \in z} \exp(-\pi k^2 \xi + i2\pi k\eta)$, Jacobi theta function [5]; $\Delta_\perp = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$, transverse Laplacian; ζ_i, random measurement error; $S_{n,\alpha}(t)$, smoothing cubic spline; $\Delta^2(t)$, "dispersion" of the measurement error; $D(q_{n,\alpha})$, root mean square measurement error.

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